<u>Announcements</u>: • HW#1 is back

• HW#3 due Fri Jan 25th (there was a typo on website)

-Reminder: PUT YOUR NAME ON YOUR HOMEWORK!!

• Midterm: $\boxed{\text{Monday Feb 11, 6:30PM}}$

- If you have a conflict, let me know.

<u>Warmup</u>: Negate the following statement.

P: "For every integer n, (if n is congruent to 3 (mod 4) then for any integers a and b, $a^2+b^2 \neq n$.)"

$$P: \forall n \in \mathbb{Z}, (n \equiv 3 \pmod 4 \implies \forall a,b \in \mathbb{Z}, (a^2+b^2 \neq n))$$

$$\sim P: \sim \forall n \in \mathbb{Z}, (n \equiv 3 \pmod 4) \Rightarrow \forall a,b \in \mathbb{Z}, (a^2+b^2 \neq n))$$

$$\equiv \exists n \in \mathbb{Z}, (n \equiv 3 \pmod 4) \not\Rightarrow \forall a,b \in \mathbb{Z}, (a^2+b^2 \neq n))$$

$$\equiv \exists n \in \mathbb{Z}, (n \equiv 3 \pmod 4) \text{ and } \sim \forall a,b \in \mathbb{Z}, (a^2+b^2 \neq n))$$

$$\equiv \exists n \in \mathbb{Z}, (n \equiv 3 \pmod 4) \text{ and } \exists a,b \in \mathbb{Z}, (a^2+b^2 = n)).$$

Nested Quantifiers: Ex: $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, (x<y))$

Ex: $S = \{1, 2, 3, 4, 5\}$

P: "$\forall x \in S, \exists y \in S, |x-y| \leq 1$" For each x, there exists a y s.t. $|x-y| \leq 1$ ✓

Q: "$\exists y \in S, \forall x \in S, |x-y| \leq 1$"

y \ x	1	2	3	4	5
1	✓	✓	✗	✗	✗
2	✓	✓	✓	✗	✗
3	✗	✓	✓	✓	✗
4	✗	✗	✓	✓	✓
5	✗	✗	✗	✓	✓

✓: $|x-y| \leq 1$

✗: $|x-y| \not\leq 1$

TRUE P: Every column (✗ ✓ ✗ ✗ ✗) has at least one ✓

FALSE Q: There is some row where every entry is ✓

✓ ✓ ✓ ✓ ✓

In general: $\underbrace{(\forall x \in S, \exists y \in S, R(x,y))}_{\text{Weaker}} \Longleftarrow \underbrace{(\exists y \in S, \forall x \in S, R(x,y))}_{\text{Stronger}}$

Weaker: y chosen in response to x

Stronger: y chosen universally.

Example: Consider

y is chosen in response to x.

$$\forall x \in \mathbb{R}_{>0}, \left(\exists y \in \mathbb{R}_{>0}, (y < x) \right)$$

For every positive real x — there exists positive real y

Proof: Suppose that x is any positive real number.

Then, let $y = \frac{x}{10}$

Since $\frac{9}{10}x$ is positive, it follows

$$x > \frac{x}{10} = y.$$

Thus, $y < x$. We have shown that for any x, there exists a y s.t. $y < x$. ∎

Comment: $\exists y \in \mathbb{R}_{>0}, \left(\forall x \in \mathbb{R}_{>0}, (y < x) \right)$ is FALSE.

__Exercise:__ For each of the listed S and $P(x,y)$, determine the truth value of $\forall x \in S(\exists y \in S(P(x,y)))$ and $\exists y \in S(\forall x \in S(P(x,y)))$

I. • $S = \mathbb{N}$, $P(x,y): x \geq y$

II. • $S = \mathbb{Z}$, $P(x,y): x \mid y$

III. • $S = \mathbb{Q}$, $P(x,y): x^2 = y$

rationals

$\left\{ \frac{a}{b} : a,b \in \mathbb{Z}, \ b \neq 0 \right\}$

• $S = \mathbb{R}$, $P(x,y): x > y$ (just discussed)

I. $\forall x \in \mathbb{N}, (\exists y \in \mathbb{N}, (x \geq y))$. True. For every x, take $y = 1$.
$\exists y \in \mathbb{N} (\forall x \in \mathbb{N}, (x \geq y))$. True, take $y = 1$. Then for every x, $x \geq 1$.

II. $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, (x \mid y))$ True. For every x, take $y = 0$.
$\exists y \in \mathbb{Z}, (\forall x \in \mathbb{Z}, (x \mid y))$. True, take $y = 0$. Then for every x, $x \mid 0$.

III. $\forall x \in \mathbb{Q}, (\exists y \in \mathbb{Q}, (x^2 = y))$ True. For every x, take $y = x^2$
$\exists y \in \mathbb{Q}, (\forall x \in \mathbb{Q}, (x^2 = y))$. False. For any y, if we pick $x = y + 2$ then
eq: $\forall y \in \mathbb{Q}, (\exists x \in \mathbb{Q}, (x^2 \neq y))$

we find $x^2 = y^2 + 4y + 4$
$> 0 + y + 0 = y$

so $x^2 > y$. Then $x^2 \neq y$.